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The collapse time of a closed cavity

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The collapse time of a closed cavity that is initially at rest in an incompressible, inviscid fluid of density ρ and ambient pressure p_∞ has the form

$$t_1 = \{\rho/(p_\infty - p_c)\}^{\frac{1}{2}} \ell,$$

where p_c is the internal pressure, which is assumed to remain constant during collapse, and ℓ is a length that depends only on the geometry of the cavity. A variational formulation of the dynamical problem is constructed from Jacobi's statement of the principle of least action. A single-degree-of-freedom approximation is developed from the similarity hypothesis that the cavity collapses through a family of similar surfaces with volume as the generalized co-ordinate. Two-degree-of-freedom approximations are given for both prolate and oblate spheroidal cavities and are used to obtain error estimates for the similarity approximation (approximately 2% for the limiting case of a needle-like, prolate spheroid and approximately $\frac{1}{2}$% for a disk-like, oblate spheroid). A perturbation analysis is developed for an approximately spherical cavity, which is found to have the same collapse time as a spherical cavity of equal volume within a factor $1 + O(e^4)$, where e is a representative eccentricity. A first-order correction for surface tension is obtained.

1. Introduction

We seek a rational approximation to the collapse time, say t_1, of a closed cavity from an initial state of rest in an incompressible, inviscid liquid of density ρ and ambient pressure p_∞. We assume that the internal pressure of the cavity, say p_c, remains constant during collapse ($0 < t < t_1$), neglect gravity, and include surface tension only through the first-order correction of the following section. We also assume that the cavity surface is simply connected and smooth (has bounded curvatures at every point); we would expect the subsequent approximations to be suited primarily to smooth, convex cavities, but convexity is not a necessary condition for the general formulation.

The motivation for our study is derived from water-entry cavities (Birkhoff & Zarantonello 1957, pp. 240–1; May 1952; May & Hoover 1965; and other references given there). The collapse sequences observed by May & Hoover, together with the measured collapse trajectories for spherical cavities (Birkhoff & Zarantonello, pp. 237, 238), suggest that our assumptions should provide a reasonable model for the estimation of collapse times of elongated axisymmetric cavities; however, the data presently available in the literature do not permit significant comparisons with our theoretical predictions of collapse times for non-spherical cavities. We emphasize that, in any event, our assumptions are certainly inadequate, and are not intended for a discussion of the terminal motion of small cavitation bubbles, which may tend to strongly asymmetric shapes (cf. Benjamin & Ellis 1965).

These last remarks notwithstanding, we present our analysis with the hope that the general formulation in §§ 3 and 4 may be of interest for other aspects of cavitation.

We infer from dimensional considerations that

$$t_1 = \{\rho/(p_\infty - p_c)\}^{\frac{1}{2}} \ell, \tag{1.1}$$

where ℓ is a length that must be proportional to the product of a characteristic scale and a function of those dimensionless parameters that are required to describe the initial shape of the cavity. A convenient scale is provided by the equivalent spherical radius R_0, as defined by

$$Q_0 = Q(0) = 4\pi R_0^3/3, \tag{1.2}$$

where $Q(t)$ is the instantaneous volume of the cavity. We then can write

$$\ell = R_0 f(e_1, e_2, \ldots), \tag{1.3}$$

where $e_1, e_2, \ldots$ are shape parameters; e.g. $f = f(e)$ for a spheroid of eccentricity e.

An explicit result appears to be known only for a sphere, for which $\ell = 0{\cdot}915 R_0$ (Lamb 1932, p. 122; Lamb refers the result to Besant 1859 and to Rayleigh 1917; Besant traces it back to the Cambridge Senate House Problems of 1847). Demtchenko (1926) and Poncin (1939*a*, *b*) have considered ellipsoidal cavities 'without obtaining any simple results' (Birkhoff & Zarantonello 1957).

The time-dependent boundary-value problem, as determined by the equations of motion for the liquid, requires the satisfaction of two, non-linear boundary conditions on the (unknown) moving surface and appears to be rather intractable for other than a spherical surface. Accordingly, we develop an approximate formulation from considerations of energy and similarity. Our approximations will be of an increasingly *ad hoc* character as we proceed from the general to the particular, but we emphasize that our general formulation, in §§ 3 and 4, yields an approximation to t_1 that should converge to the exact result as the number of degrees of freedom is increased.

The potential energy, relative to the initial configuration, is given by

$$U = (p_\infty - p_c)(Q - Q_0) \tag{1.4}$$

and is negative during the collapse. Remarking that $\dot{Q}$ must be the total volumetric flux across any surface enclosing the cavity and drawing an analogy between the cavity and a charged conductor bearing a total charge of $\dot{Q}/4\pi$, we pose the kinetic energy in the form (cf. Rayleigh 1945, § 304)

$$T = \tfrac{1}{2}\rho(M/4\pi)\dot{Q}^2, \tag{1.5}$$

where M has the dimensions of inverse length. Following electrical terminology, we designate M as the *elastance* of the cavity.

Let $q_1, q_2 \ldots, q_N$ be a set of generalized co-ordinates appropriate to the geometrical description of the instantaneous cavity and let (x_0, y_0, z_0) be the co-ordinates of its centroid.† Then T must be a homogeneous quadratic function of $\dot{x}_0, \dot{y}_0, \dot{z}_0, \dot{q}_1, \ldots, \dot{q}_N$ with coefficients that depend only on $q_1, \ldots, q_N$ and are determined by a set of kinematically specified potential problems. Invoking the assumptions that there are no external force fields and that the liquid is initially at rest, we infer from impulse-momentum considerations that $\dot{x}_0, \dot{y}_0, \dot{z}_0$ must be linear functions of $\dot{q}_1, \ldots, \dot{q}_N$, in consequence of which T, and therefore $M\dot{Q}^2$, can be expressed as a quadratic function of $\dot{q}_1, \ldots, \dot{q}_N$. We carry this calculation out in § 3.

The fact that the potential energy depends only on the instantaneous volume suggests that we choose Q or, more conveniently,

$$q = (Q/Q_0)^{\frac{1}{3}} \tag{1.6}$$

as the dominant ($i = 1$) co-ordinate. We then must choose the remaining co-ordinates to specify the instantaneous shape of the cavity e.g. in § 5 below, we choose the instantaneous eccentricity as the second co-ordinate for an ellipsoid of revolution on the hypothesis that it collapses through a family of such surfaces.

Having T and U, we could invoke Hamilton's principle to obtain a set of differential equations for $q_1, \ldots, q_N$. Actually, we find it more expedient to invoke conservation of energy at the outset and to use the resulting integral of the equations of motion to reduce the number of degrees of freedom by one, after which we regard q, rather than t, as the independent variable in terms of which the remaining variables $(q_2, \ldots, q_N)$ are to be expressed. We then can invoke Jacobi's formulation of the principle of least action to obtain either the differential equations for $q_2(q), \ldots$ or a variational integral on which to base a suitable Ritz approximation.

The energy integral, as implied by (1.4) and (1.5), is

$$E = (\rho/8\pi)\, M\dot{Q}^2 + (p_\infty - p_c)\,(Q - Q_0) = 0. \tag{1.7}$$

Integrating (1.7) between $(Q, t) = (Q_0, 0)$ and $(0, t_1)$ and comparing the result with (1.1), we obtain

$$\ell = (8\pi)^{-\frac{1}{2}} \int_0^{Q_0} \{M/(Q_0 - Q)\}^{\frac{1}{2}}\, dQ \tag{1.8a}$$

$$= 3(Q_0/8\pi)^{\frac{1}{2}} \int_0^1 q^{\frac{3}{2}}(1 - q^3)^{-\frac{1}{2}} M_1^{\frac{1}{2}}\, dq \quad (M_1 \equiv qM). \tag{1.8b}$$

To complete the calculation, we must determine M as a function of q through the procedure outlined in the preceding paragraphs.

We obtain a single-degree-of-freedom approximation to ℓ by supposing that the cavity collapses through a one-parameter family of similar surfaces, as is exactly true for a sphere. Invoking the self similarity inherent in this approximation, which we denote by an asterisk subscript, we infer that (since M is an inverse length)

$$M_{1*} = qM_* \equiv M_0, \tag{1.9}$$

† I an indebted to Dr T. B. Benjamin for pointing out the possibility of centroidal motion of the deforming cavity, even in the absence of gravitational and other, external force fields (cf. Benjamin & Ellis 1965).

where M_0 denotes the initial value of M_*. Substituting (1.2) and (1.9) into (1.8*b*), we obtain

$$
\begin{aligned}
(M_0 R_0^3)^{-\frac{1}{2}} \ell_* &= (\tfrac{3}{2})^{\frac{1}{2}} \int_0^1 q^{\frac{3}{2}}(1-q^3)^{-\frac{1}{2}}\, dq \\
&= (\tfrac{1}{6})^{\frac{1}{2}} \int_0^1 x^{-\frac{1}{6}}(1-x)^{-\frac{1}{2}}\, dx \\
&= (\tfrac{1}{6})^{\frac{1}{2}} B(\tfrac{5}{6}, \tfrac{1}{2}) = 0{\cdot}915, \qquad (1.10)
\end{aligned}
$$

where B denotes a beta function.

We demonstrate, in § 3 below, that $M_0 \geqslant 1/C_0$, where C_0 is the initial, geometric capacitance of the cavity; accordingly,

$$\ell_* \geqslant 0{\cdot}915(R_0/C_0)^{\frac{1}{2}} R_0 \qquad (1.11)$$

provides a lower bound to the approximation (1.10), although not necessarily to ℓ (see (5.20) for a counter example). We designate the right-hand side of (1.11) as the electrostatic approximation to ℓ. We also demonstrate, in § 3, that the similarity approximation implies the electrostatic approximation for an ellipsoid, for which (1.11) therefore becomes an equality. This equality does not appear to hold for more general surfaces.

We calculate two-degree-of-freedom approximations to M for prolate and oblate spheroidal cavities in §§ 5 and 6 below and then use variational approximations to obtain the corresponding results for ℓ. We surmise from these results that the similarity approximation is likely to be adequate for any smooth, convex cavity.† On the other hand, because of the identity anticipated in the preceding paragraph, the results of §§ 5 and 6 do not provide any basis for judging the electrostatic approximation.

Procedures for approximating C_0 are well developed in the literature of electricity and magnetism and can be either applied directly to obtain the electrostatic approximation to ℓ or modified to obtain M. For example, Rayleigh (1916) has shown that the capacitance of an approximately spherical surface is given by

$$C_0 = R_0[1 + O(e^4)] \quad (e \to 0), \qquad (1.12)$$

where e is a representative eccentricity, say

$$e = [1 - (b/a)^2]^{\frac{1}{2}} \qquad (1.13)$$

in terms of the minimum ($2b$) and maximum ($2a$) diameters of the surface. Similarly, we demonstrate in § 7 that

$$\ell = 0{\cdot}915 R_0[1 + O(e^4)]. \qquad (1.14)$$

We emphasize that this result involves neither the similarity nor the electrostatic approximation. Indeed, on calculating the actual coefficient of e^4 in the expansion of ℓ about $0{\cdot}915 R_0$, we find that each of the similarity and the electrostatic approximations is in error by $O(e^4)$ and therefore cannot be used to

† It might be argued that the success of the similarity approximation is at least partially a consequence of the relative insensitivity of the collapse time to the details of the motion during the final stages of collapse ($\dot{q} \to \infty$ as $q \to 0$). The force of this argument depends essentially on what is regarded as an appropriate scale; thus the integrand in (1.10) vanishes like $q^{\frac{3}{2}}$ as $q \to 0$, but is singular like $x^{-\frac{1}{6}}$ as $x \to 0$.

improve the approximation of (1.14) for an approximately spherical surface. We also remark that Poincaré's bound (Pölya & Szegö 1951, p. 17),

$$C_0 \geqslant R_0 \tag{1.15}$$

for any star-shaped surface, has no counterpart for ℓ, which can lie either above or below $0{\cdot}915R_0$ (see §5 below).

The penultimate sentence does not, of course, apply to cavities that depart widely from a sphere, with very elongated (needle-like) and very flat (disk-like) cavities as opposite extremes, and the results of §§5 and 6 provide ample support for the similarity approximation when e is not small, We also have obtained (unpublished) approximations for elongated, non-spheroidal cavities by invoking the slender-body technique that has been so widely exploited in aerodynamics, maintaining uniform validity in the neighbourhoods of the blunt ends by posing the potential problem in prolate-spheroidal co-ordinates and then letting the slenderness-ratio tend to zero (cf. Tuck 1964). This technique also could be extended to flat, non-spheroidal cavities by posing the potential problem in oblate-spheroidal co-ordinates. Our primary aim, however, is to provide a reasonably firm basis for the similarity approximation described by (1.9), (1.10), (3.22) and (3.23).

2. Surface tension

We include surface tension, say σ, in our formulation by introducing the potential energy

$$U = (p_\infty - p_c)(Q - Q_0) + \sigma(S - S_0) \tag{2.1}$$

in place of (1.4); S denotes the surface area of the cavity.

Let

$$\epsilon = \sigma S_0/(p_\infty - p_c)\, Q_0 \tag{2.2}$$

be a dimensionless measure of the relative importance of surface tension and pressure. We proceed on the hypothesis that the separate effects of surface tension and change of shape are small,

$$|\ell - \ell_*|/\ell_* \ll 1, \quad \epsilon \ll 1, \tag{2.3}$$

by virtue of which we can regard their interaction as a second-order effect and invoke the similarity approximation

$$S = S_0 q^2 \tag{2.4}$$

in determining the first-order (in ϵ) correction to ℓ.

Substituting (2.4) into (2.1) and modifying the analysis of the preceding section accordingly, we obtain

$$\ell_* = (3M_0 R_0^3/2)^{\frac{1}{2}} \int_0^1 q^{\frac{3}{2}}\{(1-q^3) + \epsilon(1-q^2)\}^{-\frac{1}{2}}\, dq. \tag{2.5}$$

This last integral can be expressed in terms of elliptic integrals, but it suffices for our purpose to remark that

$$\{(1-q^3)+\epsilon(1-q^2)\}^{-\frac{1}{2}} = (1-q^3)^{-\frac{1}{2}} \sum_0^\infty \{(-1)^n\, \Gamma(n+\tfrac{1}{2})/\Gamma(\tfrac{1}{2})\, \Gamma(n+1)\}\, \epsilon^n \left(\frac{1+q}{1+q+q^2}\right)^n \tag{2.6}$$

converges uniformly with respect to q in $q = (0, 1)$ for $\epsilon < 1$, and hence that $\ell_*(\epsilon)$ has a power-series representation as $\epsilon \to 0$. We therefore can approximate (2.5) by

$$\ell_* = (\ell_*)_{\epsilon=0}[1 - k\epsilon + O(\epsilon^2)], \tag{2.7}$$

where

$$k = \tfrac{1}{2}\int_0^1 q^{\frac{3}{2}}(1-q^2)(1-q^3)^{-\frac{3}{2}}\,dq \bigg/ \int_0^1 q^{\frac{3}{2}}(1-q^3)^{-\frac{1}{2}}\,dq = 0{\cdot}37. \tag{2.8}$$

3. The elastance

Let $q_1, q_2, \ldots, q_N$ be a set of generalized co-ordinates, as described in § 1, let $\mathbf{r}$ be a position vector with respect to the centroid of Q_0, and let $\mathbf{r} = \mathbf{r}_0(t)$ specify the centroid of $Q(t)$. Remarking that $q_1 \equiv q$ specifies the instantaneous length scale we specify the surface of the cavity, say S, by the functional relation

$$F[(\mathbf{r} - \mathbf{r}_0)/q, q_2, \ldots, q_N] = 0 \quad (\mathbf{r} \text{ in } S). \tag{3.1}$$

We emphasize that S does not determine F uniquely. The unit, outward normal to S is given by

$$\mathbf{n} = \nabla F/|\nabla F|, \tag{3.2}$$

where ∇ is the gradient operator with respect to $\mathbf{r}$.

We begin by regarding $\mathbf{r}_0$ and the q_i as functions of t (with $\dot{q}_i = dq_i/dt$, etc.) and invoking the convention that repeated indices imply summation over $i = 1, 2, \ldots, N$. Our basic assumptions, that the liquid is inviscid and that the motion develops from an initial state of rest, guarantee the existence of a velocity potential, say $\phi(\mathbf{r}, t)$, that is a harmonic function of $\mathbf{r}$ in the infinite region bounded internally by S. The requirement that no fluid cross S implies the boundary condition

$$\frac{DF}{Dt} = \dot{q}_i(\partial F/\partial q_i) + (\nabla\phi - \dot{\mathbf{r}}_0).\nabla F = 0 \quad (\mathbf{r} \text{ in } S). \tag{3.3}$$

We next introduce the similarity transformation

$$\mathbf{r}_1 = (\mathbf{r} - \mathbf{r}_0)/q, \quad \nabla_1 = \partial/\partial\mathbf{r}_1 = q\nabla, \tag{3.4a, b}$$

$$\phi(\mathbf{r}, t) = q\dot{q}\psi(\mathbf{r}_1, q), \tag{3.4c}$$

and regard $\mathbf{r}_0, q_2, \ldots, q_N$ as functions of $\mathbf{q}$, rather than t. Introducing the expansion (in which $\boldsymbol{\chi}$ is a three-component vector)

$$\psi(\mathbf{r}_1, q) = \mathbf{p}_0 \cdot \boldsymbol{\chi}(\mathbf{r}_1, q_2, \ldots, q_N) + p_i\psi_i(\mathbf{r}_1, q_2, \ldots, q_N), \tag{3.5a}$$

$$\mathbf{p}_0 = d\mathbf{r}_0/dq, \quad p_i = dq_i/dq, \tag{3.5b, c}$$

together with (3.4), into (3.3), dividing the result through by $\dot{q}|\nabla F|$, invoking (3.2), and equating coefficients of $\mathbf{p}_0$ and p_i, we can place the result in the form

$$\partial\boldsymbol{\chi}/\partial n = \mathbf{n}, \quad \partial\psi_i/\partial n = h_i \equiv -q(\partial F/\partial q_i)|\nabla_1 F|^{-1}. \tag{3.6a, b}$$

We observe that $\mathbf{n}$, as given by (3.2), is invariant under the transformation of (3.4a, b) and that, by definition, $\partial\boldsymbol{\chi}/\partial n$ and $\partial\psi_i/\partial n$ are normal derivatives at the surface S_1, where S_1 is the surface obtained from S through the scale transformation (3.4a). We also observe that

$$h_1 = -q|\nabla_1 F|^{-1}(\partial F/\partial q) = |\nabla_1 F|^{-1}\mathbf{r}_1.\nabla_1 F = \mathbf{n}.\mathbf{r}_1 \tag{3.7}$$

is the length of the perpendicular from the centroid of S_1 to the tangent plane at $\mathbf{r}_1$; and that

$$\int (\partial\psi/\partial n)\, dS_1 = \int h_1 dS_1 = 3Q_0 = 4\pi R_0^3, \tag{3.8}$$

which implies that, of the potentials χ and ψ_i, only ψ_1 contributes to the volume flux across any surface that contains S_1, i.e. only ψ_1 contains a source term in its asymptotic representation.

The kinetic energy of the irrotational motion specified by ϕ is given by

$$T = -\tfrac{1}{2}\rho \int \phi(\mathbf{n}\,.\,\nabla\phi)\, dS, \tag{3.9}$$

where the surface integral is over S. Substituting (3.4) and (3.5) into (3.9) and taking the integral over S_1 ($dS = q^2 dS_1$), we can place the result in the form

$$T = -\tfrac{1}{2}\rho q^3 \dot{q}^2 \int \psi(\partial\psi/\partial n)\, dS_1 \tag{3.10a}$$

$$= \tfrac{1}{2}\rho q^3 \dot{q}^2 (\mathbf{p}_0 \,.\, \mathbf{m}_0 \,.\, \mathbf{p}_0 + 2\mathbf{p}_0 \,.\, \mathbf{m}_i p_i + m'_{ij} p_i p_j), \tag{3.10b}$$

where

$$\mathbf{m}_0(q_2, \ldots, q_N) = -\int \mathbf{n}\chi\, dS_1 \tag{3.11}$$

is a dyadic that is proportional to the virtual mass that would characterize the cavity if it were rigid;

$$\mathbf{m}_i(q_2, \ldots, q_N) = -\tfrac{1}{2} \int \{\mathbf{n}\psi_i + \chi(\partial\psi_i/\partial n)\}\, dS_1 \tag{3.12a}$$

$$= -\int \mathbf{n}\psi_i\, dS_1 = -\int \chi(\partial\psi_i/\partial n)\, dS_1 \tag{3.12b,c}$$

is a vector that is proportional to the dipole moment of ψ_i (cf. Benjamin & Ellis 1965; also Lamb 1932, §121*a*), and the alternative forms (3.12*b*, *c*) follow from (3.12*a*) by virtue of Green's theorem; and (the prime anticipates (3.15) below)

$$m'_{ij}(q_2, \ldots, q_N) = -\int \psi_i h_j\, dS_1 = m'_{ji}. \tag{3.13}$$

Invoking the requirement that the total impulse (which is initially zero) must vanish identically in consequence of the absence of external force fields, we obtain

$$\mathbf{m}_0 \,.\, \mathbf{p}_0 + \mathbf{m}_i p_i \equiv 0. \tag{3.14}$$

Taking the scalar product of (3.14) with $\mathbf{p}_0$ and subtracting the result from (3.10), we obtain

$$T = \tfrac{1}{2}\rho q^3 \dot{q}^2 (\mathbf{p}_0 \,.\, \mathbf{m}_i p_i + m'_{ij} p_i p_j) \tag{3.15a}$$

$$\equiv \tfrac{1}{2}\rho q^3 \dot{q}^2 m_{ij} p_i p_j, \tag{3.15b}$$

where

$$m_{ij} = m'_{ij} - \mathbf{m}_i \,.\, \mathbf{m}_0^{-1} \,.\, \mathbf{m}_j \tag{3.16}$$

follows from the solution of the vector equation (3.14) through the inversion of the dyadic (essentially a square matrix) $\mathbf{m}_0$. This completes the process of ignoration of the centroidal co-ordinates (cf. Whittaker 1944, §38).

Comparing (3.15*b*) and (1.5), we obtain

$$qM = M_1 = (4\pi R_0^6)^{-1}(\mathbf{p}_0 . \mathbf{m}_i p_i + m'_{ij} p_i p_j) \tag{3.17a}$$

$$= (4\pi R_0^6)^{-1} m_{ij} p_i p_j. \tag{3.17b}$$

We also note that (3.10*a*) implies

$$M_1 = -(4\pi R_0^6)^{-1} \int \psi(\partial\psi/\partial n)\, dS_1. \tag{3.18}$$

Summarizing, we have reduced the determination of M to the following procedure: (i) determine the harmonic functions χ and ψ_i that satisfy the boundary conditions (3.6) on the surface S_1 and appropriate null conditions at infinity; (ii) calculate $\mathbf{m}_0$, $\mathbf{m}_i$, and m'_{ij} from (3.11)–(3.13); (iii) calculate $\mathbf{p}_0$ from (3.14); (iv) calculate M_1 from (3.17*a*). This problem differs from the classical problem of determining the geometric capacitance of S_1, say C_1, in that ψ is not constant on S_1. If ψ were constant on S_1, we could regard it as the potential produced by a charge distribution of surface density $(1/4\pi)$ $(\partial\psi/\partial n)$ totalling R_0^3, and conclude that $M_1 = 1/C_1$; in fact,

$$M_1 \geqslant 1/C_1 \tag{3.19}$$

by virtue of Gauss's principle that the minimum value of the right-hand side of (3.18), subject only to the constraints (3.8) and that ψ be a harmonic function, is attained for the configuration of electrostatic equilibrium (Pölya & Szegö 1951, p. 56).

We can obtain a formal upper bound to M_1 from Kelvin's minimum-energy theorem for an irrotational flow, namely

$$M_1 \leqslant (4\pi R_0^6)^{-1} \int |\mathbf{v}|^2 dV, \tag{3.20}$$

where the volume integral is taken over the domain bounded internally by S_1, and $\mathbf{v}$ is any solution to

$$\mathbf{n}.\mathbf{v} = \mathbf{n}.\mathbf{p}_0 + p_i h_i, \quad \nabla.\mathbf{v} = 0. \tag{3.21a,b}$$

The bound provided by (3.19) is of some value as an approximation, but we have not found any direct application of (3.20).

The similarity approximation invoked in § 1 is based on the assumptions that ψ depends only on the single co-ordinate q and that $\mathbf{r}_0 \equiv 0$.† Invoking these approximations in (3.6), (3.18), and (3.19), we obtain

$$\partial\psi_*/\partial n = h_1 \quad (\mathbf{r}_1 \text{ in } S_1), \tag{3.22}$$

$$M_0 \equiv M_1 = -(4\pi R_0^6)^{-1} \int \psi h_1 dS_1, \tag{2.23}$$

and

$$M_0 \geqslant 1/C_0. \tag{3.24}$$

Remarking that the boundary condition (3.22) is precisely that satisfied by the charge distribution on an ellipsoid (Jeans 1948), we infer that the similarity

† These two assumptions are independent. We could modify the similarity approximation by including the centroidal motion implied by (3.14) with $\mathbf{p}_i \equiv 0$ for $i \geqslant 2$; however, it appears unlikely that centroidal motion could be quantitatively significant for any cavity for which the assumption of a collapse through a family of similar surfaces affords an adequate approximation.

approximation implies $M_0 = 1/C_0$ for an ellipsoid. It appears most unlikely that (3.22) describes the charge distribution on surfaces other than ellipsoids, but we have not been able to obtain a general proof of this (negative) conjecture. We consider the departure of M_0 from $1/C_0$ for non-ellipsoidal cavities in § 7.

4. Variational formulation

We now proceed to a variational formulation based on the principle of least action, namely that the integral

$$\mathscr{A} = 2\int_0^{t_1} T\,dt \tag{4.1}$$

is stationary with respect to first-order variations of each of the generalized co-ordinates about the true solution of the equations of motion. Substituting T into (4.1) from (1.5) and then eliminating $\dot{Q}$ with the aid of the energy integral (1.7), we obtain

$$\mathscr{A} = (\rho/4\pi)\int_0^{Q_0} M\dot{Q}\,dQ \tag{4.2a}$$

$$= \{\rho(p_\infty - p_c)/2\pi\}^{\frac{1}{2}}\int_0^{Q_0} M^{\frac{1}{2}}(Q_0 - Q)^{\frac{1}{2}}\,dQ \tag{4.2b}$$

$$= \{\rho(p_\infty - p_c)\,Q_0^3/2\pi\}^{\frac{1}{2}}\,J, \tag{4.2c}$$

where

$$J = 3\int_0^1 q^{\frac{3}{2}}(1-q^3)^{\frac{1}{2}}\,M_1^{\frac{1}{2}}\,dq. \tag{4.3}$$

We then have the following variational problem: determine ℓ, as given by (1.8b), subject to the variational condition

$$\delta J = 0. \tag{4.4}$$

Requiring J to be stationary with respect to the independent variations $\delta q_i, i = 2, \ldots, N$, we obtain

$$\frac{d}{dq}\left\{q^{\frac{3}{2}}(1-q^3)^{\frac{1}{2}}\frac{\partial M_1^{\frac{1}{2}}}{\partial p_i}\right\} - q^{\frac{3}{2}}(1-q^3)^{\frac{1}{2}}\frac{\partial M_1^{\frac{1}{2}}}{\partial q_i} = 0 \quad (i = 2, \ldots, N). \tag{4.5}$$

We remark that the weighting function $q^{\frac{3}{2}}(1-q^3)^{\frac{1}{2}}$ vanishes at the end-points $q = 0$ and $q = 1$, by virtue of which δq_i need not vanish at these points. We also note that the assumption that the motion begins from a state of rest implies

$$\dot{q}_i = \dot{q}p_i \sim (1-q)^{\frac{1}{2}}p_i \to 0 \quad (q \to 1). \tag{4.6}$$

The differential equations (4.5) could be solved numerically (special attention would have to be paid to the singular points at $q = 0$ and $q = 1$); on the other hand, analytical solutions in terms of known functions do not appear to exist for non-trivial configurations. We therefore find it expedient, in the subsequent development, to introduce approximate representations of the form

$$q_i = q_i(q, a_i, b_i \ldots) \quad (i = 2, \ldots, N), \tag{4.7}$$

where $a_i, b_i, \ldots$ are free parameters, and to determine the optimum values of these parameters by requiring

$$\partial J/\partial a_i = \partial J/\partial b_i = \ldots = 0. \tag{4.8}$$

Note on principle of least action

Substituting M from (3.17*b*) and $E-U$ from (1.4) and (1.7) into (4.2*b*), we obtain

$$\mathscr{A} = (2\rho)^{\frac{1}{2}} \int (E-U)^{\frac{1}{2}}\, ds, \tag{4.9}$$

where

$$ds = (q^3 m_{ij}\, dq_i\, dq_j)^{\frac{1}{2}} \tag{4.10}$$

is the *kinematic line element* in the phase space $(q, q_2, \ldots)$, and $E = T+U$ is the total energy ($E \equiv 0$ in the present application). This is Jacobi's formulation of the action integral (Synge 1960, p. 139).

We also note that (4.5) could be derived from Lagrange's equations with the energy integral as a constraint to reduce by one the number of degrees of freedom (Whittaker 1944, §42) after the ignoration of $\mathbf{r}_0$. However, quite aside from the fact that we wish to appeal directly to the variational integral to obtain approximate solutions, the derivation of (4.5) from Jacobi's formulation of the principle of least action appears both more aesthetic and more direct (cf. Whittaker's derivation).

5. Prolate spheroid

We now apply the general formulation of the preceding sections to a prolate spheroidal cavity. Let R_0 be the equivalent radius defined by (1.2) and δ be the slenderness ratio (minor axis/major axis) of the initial configuration; then the initial values of the major and minor semi-axes and of the eccentricity are given by

$$a = \delta^{-\frac{2}{3}} R_0, \quad b = \delta^{\frac{1}{3}} R_0, \quad e = (1-\delta^2)^{\frac{1}{2}}. \tag{5.1}$$

We invoke the two-degree-of-freedom approximation that the cavity collapses through a family of prolate spheroids with fixed centroid ($\mathbf{r}_0 \equiv 0$ by virtue of symmetry), which can be specified by two generalized co-ordinates. It is conceivable that the cavity might pass through a spherical shape to a family of oblate spheroids; however, this contingency is covered by the mathematical formulation and would be signalled by a zero of the eccentricity.

We begin by introducing the spheroidal co-ordinates ζ and μ according to

$$z = c\zeta\mu, \quad \varpi = c(\zeta^2-1)^{\frac{1}{2}}(1-\mu^2)^{\frac{1}{2}}, \tag{5.2}$$

where z and ϖ are cylindrical polar co-ordinates and c is a time-dependent length. Let ξ be the inverse eccentricity of S; then $\zeta = \xi$ on S. The instantaneous volume is given by

$$Q(t) = (4\pi/3)\, c^3 \xi(\xi^2-1) \equiv (4\pi/3)\,(R_0 q)^3, \tag{5.3}$$

which implies

$$c = c(q,\xi) = R_0 q^{-\frac{1}{3}} (\xi^2-1)^{-\frac{1}{3}}. \tag{5.4}$$

Choosing our two generalized co-ordinates as q and ξ and referring to (3.1), we define

$$F(\mathbf{r}_1, q, \xi) + 1 = R_0^{-2}\{(1-\xi^{-2})^{\frac{2}{3}} z_1^2 + (1-\xi^{-2})^{-\frac{1}{3}} \varpi_1^2\} \tag{5.5a}$$

$$= (\zeta/\xi)^2 \mu^2 + (\xi^2-1)^{-1}(\zeta^2-1)(1-\mu^2), \tag{5.5b}$$

where $z_1 = z/q$ and $\varpi_1 = \varpi/q$.

Invoking the general formulation of §3 (we emphasize that the partial derivatives $\partial F/\partial q$ and $\partial F/\partial \xi$ that enter this general formulation must be evaluated with z and ϖ, not ζ and μ, fixed), we reduce (3.6) to

$$\psi_\zeta|_{\zeta=\xi} = R_0^2 \xi^{\frac{1}{3}}(\xi^2-1)^{-\frac{2}{3}}[1-2\kappa P_2(\mu)], \tag{5.6}$$

where $P_2(\mu)$ is a Legendre polynomial, and

$$\kappa = \tfrac{1}{3}q\xi^{-1}(\xi^2-1)^{-1}(d\xi/dq) \tag{5.7a}$$

$$= \tfrac{1}{3}d\log(1-\xi^{-2})^{\frac{1}{2}}/d\log q. \tag{5.7b}$$

We observe that $(1-\xi^{-2})^{\frac{1}{2}}$ is the instantaneous slenderness ratio. Recalling that the most general solution to Laplace's equation that satisfies a null condition at $\zeta = \infty$ has the form

$$\psi = \sum_0^\infty A_n Q_n(\zeta) P_n(\mu), \tag{5.8}$$

we obtain $$\psi = R_0^2 \xi^{\frac{1}{3}}(\xi^2-1)^{-\frac{2}{3}}[\{Q_0(\zeta)/Q_0'(\xi)\} - 2\kappa\{Q_2(\zeta)/Q_2'(\xi)\}P_2(\mu)] \tag{5.9}$$

as the solution implied by the boundary condition (5.6).

Substituting (5.6) and (5.9) into (3.18), we obtain

$$M_1 = (4\pi R_0^6)^{-1}(2\pi R_0^5)\int_{-1}^{1}\{g_0(\xi) - 2\kappa g_2(\xi)P_2(\mu)\}\{1-2\kappa P_2(\mu)\}\,d\mu \tag{5.10a}$$

$$= R_0^{-1}\{g_0(\xi) + \tfrac{4}{5}\kappa^2 g_2(\xi)\}, \tag{5.10b}$$

where $$g_n(\xi) = -\xi^{\frac{1}{3}}(\xi^2-1)^{-\frac{4}{3}}Q_n(\xi)/Q'(\xi). \tag{5.11}$$

We can infer from the known properties of $Q_n(\xi)$ (Hobson 1931) that $g_n(\xi)$ has no zeros in a complex plane cut along $(-\infty, +1)$.

Setting $\xi \equiv 1/e$ and $\kappa = 0$ in (5.10b), we obtain the similarity approximations

$$R_0 M_{1*} = R_0/C_0 = \tfrac{1}{2}e^{-1}(1-e^2)^{\frac{1}{3}}\log\{(1+e)/(1-e)\} \tag{5.12a}$$

$$= 1 - \tfrac{1}{45}e^4 + O(e^6) \quad (e \to 0) \tag{5.12b}$$

$$= \delta^{\frac{2}{3}}\log(2/\delta)\{1+O(\delta^2)\} \quad (\delta \to 0) \tag{5.12c}$$

and $$\ell_* = 0{\cdot}915(R_0/c_0)^{\frac{1}{2}}R_0. \tag{5.13}$$

To obtain a two-degree-of-freedom approximation to ℓ, we must determine $\xi(q)$. The differential equation obtained by substituting (5.10) into (4.5) is intractable, but we have carried out a variational approximation based on the trial function

$$\tfrac{1}{2}\log\{(1-e^2)/(1-\xi^{-2})\} = k(1-q^3), \tag{5.14}$$

with k determined by the minimization of J. The calculation is lengthy, and the result for ℓ is a rather complicated, albeit algebraic, function of e; accordingly, we present only the limiting approximations (below) and the intermediate point

$$\ell = 1{\cdot}032\ell_* = 1{\cdot}027(0{\cdot}915R_0) \quad \text{at} \quad e = 0{\cdot}8. \tag{5.15}$$

The limiting result as $e \to 0$ is especially simple and yields

$$\ell = \ell_*\{1 + 0{\cdot}0944e^4 + 0{\cdot}0274e^6 + O(e^8)\} \tag{5.16a}$$

$$= 0{\cdot}915R_0\{1 + 0{\cdot}0833e^4 + 0{\cdot}0108e^6 + O(e^8)\}. \tag{5.16b}$$

We conclude that $0{\cdot}915R_0$ provides an excellent approximation to ℓ for sufficiently small e and that ℓ approaches both ℓ_* and $0{\cdot}915R_0$ from above as $e \to 0$. The terms in e^4 and e^6 in (5.16b) are less than 0·01 and 0·0005, respectively, for $e < 0{\cdot}59$.

The limiting case $e \to 1$ yields

$$\ell = \ell_*\{1 - 0{\cdot}027(1-\tfrac{3}{2}Q_0^{-1})^2(1-2{\cdot}25Q_0^{-1}-1{\cdot}12Q_0^{-2})(1-1{\cdot}89Q_0^{-1}-0{\cdot}59Q_0^{-2})^{-2} + O(\delta^2)\} \quad (\delta \to 0), \tag{5.17}$$

where
$$\ell_* = 0{\cdot}915\delta^{\frac{1}{3}}R_0\{\log(2/\delta)\}^{\frac{1}{2}}\{1+O(\delta^2)\} \tag{5.18}$$

and
$$Q_0 = Q_0(1/e) = \tfrac{1}{2}\log\{(1+e)/(1-e)\} \tag{5.19a}$$
$$= \log(2\delta) + O(\delta^2). \tag{5.19b}$$

We also note the limiting approximation

$$\ell \to 0{\cdot}973\ell_* \quad \{\log(2/\delta) \to \infty\}. \tag{5.20}$$

We recall, (5.1), that $\delta^{\frac{1}{3}}R_0 = b$ is the initial value of the minor semi-axis. The correction provided by (5.20) is not accurate for realistic values of δ—e.g. (5.17) yields $\ell = 0{\cdot}984\ell_*$ for $\delta = 0{\cdot}1$—but it does reveal that $\ell < \ell_*$ as $e \to 1$, in contrast to the result for $e \to 0$.

We remark that the logarithmic singularity in both ℓ and ℓ_* as $\delta \to 0$ is typical of any slender cavity. It follows that two-dimensional analysis, such as that given by Poncin (1932) for the collapse of a cylindrical cavity of elliptic cross-section, must yield infinite collapse times. The difficulty arises from the well known fact that the inertia of a two-dimensional, radial flow is infinite (cf. Benjamin 1964). There may be contexts in which this difficulty is of only secondary importance (as in Benjamin's discussion), but it clearly is crucial in the present context (cf. the difficulty of calculating the supersonic drag on a slender body of revolution).

6. Oblate spheroid

We consider next an oblate spheroidal cavity, following closely the analysis of the preceding section. Let δ be the ratio of minor to major axis, as before (but now *flatness ratio* is a more appropriate term); then

$$a = \delta^{-\frac{1}{3}}R_0, \quad b = \delta^{\frac{2}{3}}R_0, \quad e = (1-\delta^2)^{\frac{1}{2}}. \tag{6.1}$$

Introducing the oblate ellipsoidal co-ordinates ζ and μ according to

$$z = c\zeta\mu, \quad \varpi = c(1+\zeta^2)^{\frac{1}{2}}(1-\mu^2)^{\frac{1}{2}}, \quad c = R_0 q\xi^{-\frac{1}{3}}(1+\xi^2)^{-\frac{1}{3}}, \tag{6.2}$$

where $\zeta = \xi$ on S, we find that M_1 is given by (5.10b) if only (5.7) and (5.11) are replaced by

$$\kappa = \tfrac{1}{3}d\log(1+\xi^{-2})^{\frac{1}{2}}/d\log q \tag{6.3}$$

and
$$g_n(\xi) = i\xi^{\frac{1}{3}}(1+\xi^2)^{-\frac{2}{3}}Q_n'(i\xi)/Q_n(i\xi), \quad i = (-1)^{\frac{1}{2}}. \tag{6.4†}$$

We also find that the instantaneous eccentricity of the oblate ellipsoid is given by $(1+\xi^2)^{-\frac{1}{2}}$, so that

$$\xi_0 = (e^{-2}-1)^{\frac{1}{2}} = \delta/e. \tag{6.5}$$

† This is the only point in our analysis at which i denotes the imaginary unit.

We consider further only the limiting cases $e \to 0$ and $e \to 1$. We find that (5.16a, b) continue to hold (within the same variational approximation) in the former limit through $O(e^4)$ by virtue of the fact that the asymptotic approximations to the respective results for M_1 are identical through terms of $O(\xi^{-4})$ as $\xi \to \infty$. See also (7.28) below.

Letting $\xi \to 0$, we obtain

$$M_1 = (\pi/2R_0)\,\xi^{\frac{1}{3}}\left\{1+\frac{1}{45}\left(\frac{d\log\xi}{d\log q}\right)^2\right\}\{1+O(\xi)\}. \tag{6·6}$$

The corresponding approximation to the initial condition on ξ is

$$\xi = \delta \quad (q = 1,\ \delta \to 0). \tag{6.7}$$

These last results suggest the trial function (at this point, our analysis departs from that of §5)

$$\xi = \delta q^{3k}. \tag{6.8}$$

Substituting (6.6) and (6.8) into (4.3), we obtain

$$J = (\pi/2R_0)^{\frac{1}{2}}\,\delta^{\frac{1}{6}}(1+\tfrac{1}{5}k^2)^{\frac{1}{2}}\,B(\tfrac{3}{2},\tfrac{5}{6}+\tfrac{1}{6}k). \tag{6.9}$$

Equating $\partial J/\partial k$ to zero, we obtain

$$\tfrac{6}{5}k+(1+\tfrac{1}{5}k^2)\,\{\psi(\tfrac{5}{6}+\tfrac{1}{6}k)-\psi(\tfrac{7}{3}+\tfrac{1}{6}k)\} = 0, \tag{6.10}$$

where ψ is the logarithmic derivative of the gamma function. Solving (6.10) numerically, we obtain $k = 1{\cdot}27$. Substituting this result, together with (6.6) and (6.8), into (1.8b) we obtain

$$\ell = 1{\cdot}004\,\ell_* \quad (\delta \to 0), \tag{6.11}$$

where

$$\ell_* = 0{\cdot}915(\pi/2)^{\frac{1}{2}}\,\delta^{\frac{1}{2}}R_0 = 0{\cdot}915(\pi R_0^3/2a)^{\frac{1}{2}} \quad (\delta \to 0). \tag{6.12}$$

We conclude that ℓ_* provides an approximation to ℓ with an error of about 0·4 % in the limiting case of a disk-shaped cavity. This compares with the error of about 2·7 % for the limiting case of a needle-shaped cavity (see (5.20)).

7. Approximately spherical cavity

The analysis of an approximately spherical cavity is expedited by the introduction of spherical harmonics. We consider the surface of revolution specified by

$$F(\mathbf{r}_1, q_2, \ldots, q_N) = r_1^3 - R_0^3\Lambda(\mu) \tag{7.1}$$

and

$$\Lambda(\mu) = 1 + a_i P_i(\mu), \quad \mu = \cos\theta, \tag{7.2}$$

where F, $\mathbf{r}_1$, and q_i are defined in § 3, θ is the polar angle between $\mathbf{r}_1$ and the axis of symmetry, and $P_i(\mu)$ is a Legendre polynomial of order i. The assumption of axial symmetry is not essential; see (7.29) *et seq.*

We suppose that

$$a_1 = O(e^4), \quad q_i \equiv a_i = O(e^2) \quad (i = 2, \ldots, N), \tag{7.3a, b}$$

where e is the eccentricity of (1.13). The constraint that the volume of S_1 be $4\pi R_0^3/3$,

$$Q = \tfrac{2}{3}\pi\int_{-1}^{1} r^3\,d\mu = 2\pi R_0^3\int_{-1}^{1}\Lambda\,d\mu \tag{7.4}$$

independently of $q_2, \ldots, q_N$, is satisfied exactly by our choice of the leading term in the Fourier–Legendre expansion (7.2). The constraint that the centroid be at the origin is satisfied approximately by virtue of (7.3*a*), which implies that

$$\int_{-1}^{1} (r_1\mu)\, r_1^3\, d\mu = R_0^4 \int_{-1}^{1} \mu \Lambda^{\frac{4}{3}}(\mu)\, d\mu = O(e^4 R_0^4). \tag{7.5}$$

Referring to (3.5) and (3.6) and anticipating that

$$p_0 = O(e^4 R_0), \tag{7.6}$$

we write the boundary condition on ψ in the form

$$|\nabla F|\,(\partial\psi/\partial n) \equiv \nabla_1 F\,.\,\nabla_1\psi = \mathbf{r}_1\,.\,\nabla_1 F - qp_i(\partial F/\partial q_i) \tag{7.7a}$$

$$= 3r_1^3 + R_0^3 qp_i(\partial\Lambda/\partial q_i) \tag{7.7b}$$

$$= 3R_0^3\{1 + (q_i + \tfrac{1}{3}qp_i)\,P_i + O(e)^4\}, \tag{7.7c}$$

where $i = 1$ is omitted in the summation of (7.7*c*). Invoking the constraint (3.8), which implies that the mean value of ψ over any spherical surface that contains S_1 must be $-R_0^3/r_1$, we expand ψ in spherical-harmonics according to

$$\psi = -(R_0^3/r_1)\{1 + A_i(R_0/r_1)^i P_i(\mu)\}. \tag{7.8}$$

Forming the scalar product $\nabla_1 F\,.\,\nabla_1\psi$ from (7.1) and (7.8), and setting $r_1 = R_0\Lambda^{\frac{1}{3}}$ in the result, we obtain

$$\nabla_1 F\,.\,\nabla_1\psi = (\partial F/\partial r_1)(\partial\psi/\partial r_1) + r_1^{-2}(1-\mu^2)(\partial F/\partial\mu)(\partial\psi/\partial\mu) \tag{7.9a}$$

$$= 3R_0^3[1 + (i+1)\,A_i P_i(\mu)\{1 + O(e^2)\}]. \tag{7.9b}$$

Comparing (7.7*c*) and (7.9*b*) and invoking (7.3) and the orthogonality of the Legendre polynomials over $\mu = (-1, 1)$, we obtain

$$A_1 = O(e^4), \quad A_i = (i+1)^{-1}(q_i + \tfrac{1}{3}qp_i)\{1 + O(e^2)\} \quad (i = 2, \ldots, N). \tag{7.10a, b}$$

Remarking that the dipole moment of ψ is proportional to A_1 and referring to the statement immediately following (3.12), we infer (7.6) from (7.10*a*).

Substituting ψ from (7.8), $\partial\psi/\partial n$ from (7.7*c*), and

$$dS_1 = \tfrac{2}{3}\pi|\nabla_1 F|\,d\mu \tag{7.11}$$

into (3.10), we obtain

$$R_0 M_1 = -(4\pi R_0^5)^{-1}\int \psi(\partial\psi/\partial n)\, dS_1 \tag{7.12a}$$

$$= \tfrac{1}{2}\int_{-1}^{1} (R_0/r_1)\{1 + A_i(R_0/r_1)^i P_i(\mu)\}\{1 + (q_i + \tfrac{1}{3}qp_i)\,P_i(\mu)\}\,d\mu \tag{7.12b}$$

$$= \tfrac{1}{2}\int_{-1}^{1} \{1 - \tfrac{1}{3}q_i P_i + \tfrac{2}{9}(q_i q_j P_i P_j)\}\{1 + A_i P_i(1 - \tfrac{1}{3}iq_j P_j)\} \times \{1 + (q_i + \tfrac{1}{3}qp_i)\,P_i\}\,d\mu + O(e^6). \tag{7.12c}$$

Invoking the orthogonality relation

$$\tfrac{1}{2}\int_{-1}^{1} P_i P_j\, d\mu = (2i+1)^{-1}\delta_{ij}, \tag{7.13}$$

we reduce (7.12*c*) to

$$M_1 = \frac{1}{C_1} + \frac{[(i-2)\,q_i - qp_i]^2}{9(i+1)\,(2i+1)\,R_0} + O(e^6), \tag{7.14}$$

where

$$\frac{1}{C_1} = \frac{1}{R_0}\left\{1 - \frac{1}{9}\left(\frac{i-1}{2i+1}\right) q_i^2 + O(e^6)\right\} \tag{7.15}$$

is the corresponding approximation to the inverse capacitance of S_1 (Rayleigh 1916). We remark that each of the series in (7.14) and (7.15) is positive definite, as required by (3.9) and (1.15), respectively. The summations in (7.12) and throughout the remainder of this section are over $i = 2, \ldots, N$.

Substituting (7.14) into (1.8*b*) and (4.3) and introducing $x = q^3$, we obtain

$$\ell = (\tfrac{1}{6})^{\frac{1}{2}} R_0 \int_0^1 x^{-\frac{1}{6}}(1-x)^{-\frac{1}{2}}\{1 + (i+1)^{-1}(2i+1)^{-1} f_i(x) + O(e^6)\}^{\frac{1}{2}}\, dx \tag{7.16a}$$

$$= R_0\{0{\cdot}915 + \tfrac{1}{2}(\tfrac{1}{6})^{\frac{1}{2}}(i+1)^{-1}(2i+1)^{-1} I_i + O(e^6)\} \tag{7.16b}$$

$$= 0{\cdot}915 R_0\{1 + O(e^4)\}, \tag{7.16c}$$

and

$$R_0^{\frac{1}{2}} J = \int_0^1 x^{-\frac{1}{6}}(1-x)^{\frac{1}{2}}\{1 + (i+1)^{-1}(2i+1)^{-1} f_i(x) + O(e^6)\}^{\frac{1}{2}}\, dx \tag{7.17a}$$

$$= B(\tfrac{5}{6}, \tfrac{3}{2}) + \tfrac{1}{2}(i+1)^{-1}(2i+1)^{-1} J_i + O(e^6), \tag{7.17b}$$

where

$$f_i(x) = (x\,dq_i/dx)^2 - \tfrac{2}{3}(i-2)\,xq_i(dq_i/dx) - \left(\frac{4i-5}{9}\right) q_i^2 \quad (i = 2, \ldots, N), \tag{7.18}$$

$$I_i = \int_0^1 x^{-\frac{1}{6}}(1-x)^{-\frac{1}{2}} f_i(x)\, dx, \tag{7.19}$$

and

$$J_i = \int_0^1 x^{-\frac{1}{6}}(1-x)^{\frac{1}{2}} f_i(x)\, dx. \tag{7.20}$$

The approximations (7·16*b*) and (7.17*b*) follow only after interchanging the order of expansion of the integrands in e^2 and integration with respect to x and are, therefore, provisional (see below).

We consider briefly the behaviour of $q_i(q)$, as determined by the Euler equations of (4.5); our aim is to illustrate the implications of the singular points at $q = 0$ and $q = 1$, rather than to provide a direct basis for numerical computation. Substituting M_1 from (7.14) and (7.15) into (4.5), introducing $x = q^3$, and linearizing in q_i (or, equivalently, forming the Euler equations implied by the requirement that (7.17*b*) be stationary), we obtain

$$x^2(1-x)\frac{d^2q_i}{dx^2} + (\tfrac{11}{6} - \tfrac{7}{3}x)\,x\frac{dq_i}{dx} + (\tfrac{1}{6}i - \tfrac{1}{3}x)\,q_i = 0 \quad (i = 2, \ldots, N). \tag{7.21}$$

Transforming (7.21) to the hypergeometric equation and invoking the initial conditions of (4.6), we obtain (we omit a considerable amount of analytical detail throughout the present paragraph)

$$q_i(q)/q_i(1) = x^{-(\frac{5}{12})+\nu} F(\tfrac{7}{12} + \nu, -\tfrac{1}{12} + \nu; \tfrac{1}{2}; 1-x) \tag{7.22a}$$

$$= \mathrm{Re}\{[\Gamma(\tfrac{1}{2})\,\Gamma(-2\nu)/\Gamma(\tfrac{7}{12} - \nu)\,\Gamma(-\tfrac{1}{12} - \nu)]\, x^{-(\frac{5}{12})+\nu} \times F(\tfrac{7}{12} + \nu, -\tfrac{1}{12} + \nu; 1 + 2\nu; x)\}, \tag{7.22b}$$

where

$$\nu = \tfrac{1}{12}(25 - 24i)^{\frac{1}{2}} \tag{7.23}$$

is an imaginary number (for $i \geqslant 2$), F is a hypergeometric function, and Re{} is the real part of {}. We infer from (7.22*a*) that the solution for $q_i(x)$ implied by the antecedent approximations—in particular, the linearization of the Euler equations—is regular, and therefore uniformly valid, in the neighbourhood of the singular point at $x = 1$ $(t = 0)$. On the other hand, we infer from (7.22*b*) that this solution is not uniformly valid in the neighbourhood of $x = 0$; indeed, substituting (7.22*b*) into (7.18), we find that

$$f_i(x) = O[x^{-\frac{5}{6}} \cos(2|\nu| \log x + C)] \quad (x \to 0), \tag{7.24}$$

where C is a real constant. Substituting (7.24) into (7.19) and (7.20), we obtain divergent, although finite, integrals (we could circumvent these divergences by the artifice of replacing the exponent $-\frac{5}{12}$ in (7.22) by $-a$ and letting $a \to (\frac{5}{12})-$ after carrying out the integrations, but, even then, the integrals are intractable).

The growth of small perturbations of a spherical cavity has been calculated by Plesset & Mitchell (1955), and our (7.22) is equivalent to their (29); see also Birkhoff & Zarantonello (1957, p. 253). This predicted instability is quite weak and may be eliminated by viscosity; in any event, it does not appear to have any substantial effect on the collapse times of a spherical bubble (Birkhoff & Zarantonello 1957, p. 237), presumably in consequence of the rapidity of the final portion of the collapse trajectory.

The preceding results suggest that suitable trial functions for direct, variational approximations can be constructed as expansions about $x = 1$, even though such expansions may not be uniformly valid near $x = 0$. The exact behaviour of q_i as $x \to 1$ is given by (7.22*a*) as

$$q_i(q)/q_i(1) = 1 - \tfrac{1}{3}(i-2)(1-x) + O(1-x)^2 \quad (x \to 1); \tag{7.25}$$

this suggests that the simplest trial function should be linear for $i > 2$ but quadratic for $i = 2$. The integrals obtained by substituting such trial functions in (7.18)–(7.20) can be expressed in terms of beta functions and are algebraic in the coefficients of the trial functions (cf. §§ 5 and 6). We emphasize that there is no reason to expect that these coefficients, as determined by the requirement $\delta J = 0$, should agree with these given by the power-series expansion indicated by (7.25); the variational approximation aims at fitting the solution over the entire interval, $x = (0, 1)$, whereas truncation of the true power-series solution typically yields a good approximation only near $x = 1$.

Let us consider the prolate spheroid of § 5, which is specified by

$$r = R_0(1-\xi^{-2})^{\frac{1}{6}}(1-\xi^{-2}\mu^2)^{-\frac{1}{2}} \tag{7.26}$$

in the spherical polar co-ordinates of the present section (we emphasize that μ is now defined differently than in § 5, but $1/\xi$ is still the instantaneous eccentricity). Expanding (7.26) in Legendre polynomials, we obtain

$$q_2 = \xi^{-2}, \quad q_i = O(\xi^{-4}) \quad (i > 2). \tag{7.27a, b}$$

Substituting (7.27) into (7.14) and (7.15), we obtain

$$R_0 M_1 = 1 - \tfrac{1}{45}\xi^{-4} + \frac{1}{135}\left(q\frac{d}{dq}\xi^{-2}\right)^2 + O(\xi^{-6}), \tag{7.28}$$

in agreement with (5.10*b*) through terms of $O(\xi^{-4})$. The corresponding approximation implied by (5.14) and (7.16)–(7.20) reduces to (5.16*b*).

To generalize our results to an asymmetric cavity, we need only replace $P_i(\mu)$ in (7.2) *et seq.* by the more general surface harmonic S_i and $(2i+1)^{-1}$ in (7.14) and (7.15) by the mean-square value of S_i. We can bring our expansions into standard form (for spherical surface harmonics) by modifying our notation for the generalized co-ordinates according to

$$\Lambda = 1+\sum_{i=2}^{N}\left[q_i P_i(\mu)+\sum_{j=1}^{i} P_i^j(\mu)\{q_{ij}^{(e)}\cos(j\phi)+q_{ij}^{(0)}\sin(j\phi)\}\right], \tag{7.29}$$

where the total number of generalized co-ordinates—$q, q_2, \ldots, q_N, q_{21}^{(e)}, \ldots, q_{NN}^{(0)}$—is now

$$1+(N-1)+2\sum_{i=2}^{N} i = (N+1)^2-3, \tag{7.30}$$

rather than N, and the three surface harmonics $P_1(\mu)$, $P_1^1(\mu)\cos\phi$, and $P_1^1(\mu)\sin\phi$ are omitted in order to place the centroid at the origin (with an error of $O(e^4)$). We find that

$$\frac{1}{18R_0}\sum_{i=2}^{N}\sum_{j=1}^{i}\frac{(i+1)!\,\{(i-2)\,q_{ij}^{(e)}-p_{ij}^{(e)}\}^2}{(i-j)!\,(i+1)\,(2i+1)}, \tag{7.14a}$$

$$\frac{1}{18R_0}\sum_{i=2}^{N}\sum_{j=1}^{i}\frac{(i+j)!\,(i-1)\,\{q_{ij}^{(e)}\}^2}{(i-j)!\,(2i+1)}, \tag{7.15a}$$

and similar series in $q_{ij}^{(0)}$, must be added to the right-hand sides of (7.14) and (7.15). The corresponding generalizations of (7.16)–(7.20) are obvious.

8. Conclusion

We conclude that the similarity approximation (1.10), with M_0 calculated according to (3.22) and (3.23), is likely to be adequate for most applications in which our basic assumptions (stated and discussed in the opening paragraph of § 1 above) can be regarded as valid.

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REFERENCES

Benjamin, T. B. 1964 *J. Fluid Mech.* **19**, 137–144.

Benjamin, T. B. & Ellis, A. 1965 *Proc. Roy. Soc.* A (in the Press.)

Besant, H. 1859 *Hydrostatics and Hydrodynamics.* Cambridge University Press.

Birkhoff, G. & Zarantonello, E. 1957 *Jets, Wakes and Cavities.* New York: Academic Press.

Demtchenko, B. 1926 *J. l'École Polytechnique*, **27**, 113–121.

Hobson, E. W. 1931 *Spherical and Ellipsoidal Harmonics.* Cambridge University Press.

Jeans, J. 1948 *The Mathematical Theory of Electricity and Magnetism*, p. 247. Cambridge University Press.

Lamb, H. 1932 *Hydrodynamics.* Cambridge University Press.

May, A. 1952 *J. Appl. Phys.* **23**, 1362–72.

May, A. & Hoover, W. R. 1965 A Study of the Water-Entry Cavity. *Ballistics Res. Rep.* no. 121, U.S. Naval Ord. Lab., White Oak, Maryland.

PLESSET, M. S. & MITCHELL, T. P. 1955 *Quart. Appl. Math.* **13**, 419–430.

PÓLYA, G. & SZEGÖ, G. 1951 *Isoperimetric Inequalities in Mathematical Physics*. Princeton University Press.

PONCIN, H. 1932 *Publ. Sci. et Tech. du Ministere de l'Air*, no. 18, Paris.

PONCIN, H. 1939*a* *J. Math. Pures Appl.* **18**, 385–404.

PONCIN, H. 1939*b* *Acta Math.* **71**, 1–62.

RAYLEIGH, LORD 1916 *Phil. Mag.* **31**, 177–186; *Scientific Papers*, **6**, 383–392.

RAYLEIGH, LORD 1917 *Phil. Mag.* **34**, 94–98; *Scientific Papers*, **6**, 504–507.

RAYLEIGH, LORD 1945 *Theory of Sound*. New York: Dover.

SYNGE, J. L. 1960 Classical Dynamics. *Principles of Classical Mechanics and Field Theory*, vol. III of *Encyclopedia of Physics*. Berlin: Springer-Verlag.

TUCK, E. O. 1964 *J. Fluid Mech.* **18**, 619–635.

WHITTAKER, E. T. 1944 *Analytical Dynamics*. New York: Dover.